Perturbation analysis of $A^{(2)}_{T,S}$ on Hilbert spaces

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Abstract

In this paper, we investigate the perturbation analysis of $A^{(2)}_{T,S}$ when T, S and A have some small perturbations on Hilbert spaces. We present the conditions that make the perturbation of $A^{(2)}_{T,S}$ is stable. The explicit representation for the perturbation of $A^{(2)}_{T,S}$ and the perturbation bounds are also obtained.

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1 Introduction

Let X, Y be Banach spaces and let $B(X,Y)$ denotes the set of bounded linear operators from X to Y. For an operator $A \in B(X,Y)$, let $R(A)$ and $N(A)$ denote the range and kernel of A, respectively. Let T be a closed subspace of X and S be a closed subspace of Y. Recall that $A^{(2)}_{T,S}$ is the unique operator G satisfying

$$GAG = G, \quad R(G) = T, \quad N(G) = S. \tag{1.1}$$

It is known that (1.1) is equivalent to the following condition:

$$N(A) \cap T = \{0\}, \quad AT \dotplus S = Y \tag{1.2}$$

(cf. [5, 6]). It is well–known that the commonly five kinds of generalized inverse: the Moore–Penrose inverse A^+, the weighted Moore–Penrose inverse A^+_{MN}, the Drazin inverse A^D, the group inverse $A^\#$ and the Bott–Duffin inverse $A^{(-1)}_{(L)}$ can be reduced to a $A^{(2)}_{T,S}$ for certain choices of T and S.

The perturbation analysis of $A^{(2)}_{T,S}$ have been studied by several authors (see [12, 13], [16, 17]) when X and Y are of finite–dimensional. A lot of results about the error bounds have been obtained. When X and Y are of infinite–dimensional Banach spaces, the perturbation analysis of $A^{(2)}_{T,S}$ for small perturbation of T, S and A has been done in [7].

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In this paper, we assume that X and Y are all Hilbert spaces over the complex field $\mathbb{C}$. Using the theory of stable perturbation of generalized inverses established by G. Chen and Y. Xue in [2, 3], we will give the upper bounds of $\|\bar{A}^{(2)}_{T',S'}\|$ and $\|\bar{A}^{(2)}_{T',S'} - A^{(2)}_{T,S}\|$ respectively for certain T', S' and $\bar{A}$. The results in this paper improve [14, Theorem 4.4.7].

2 Preliminaries

Let H be a complex Hilbert space. Let V be a closed subspace of H. We denote by P_V the orthogonal projection of H onto V. Let M, N be two closed subspaces in H. Set

$$\delta(M,N) = \begin{cases} \sup\{dist(x,N)\,|\, x\in M,\ \|x\|=1\}, & M\neq\{0\} \\ 0 & M=\{0\} \end{cases},$$

where $dist(x,N) = \inf\{\|x-y\|\,|\,y\in N\}$. The gap $\hat{\delta}(M,N)$ of M, N is given by $\hat{\delta}(M,N) = \max\{\delta(M,N),\delta(N,M)\}$. For convenience, we list some properties about $\delta(M,N)$ and $\hat{\delta}(M,N)$ which come from [9] as follows.

Proposition 2.1 ([9]). *Let M, N be closed subspaces in a Hilbert space H.*

(1) $\delta(M,N)=0$ *if and only if* $M\subset N$

(2) $\hat{\delta}(M,N)=0$ *if and only if* $M=N$

(3) $\hat{\delta}(M,N)=\hat{\delta}(N,M)$

(4) $0\le\delta(M,N)\le 1$, $0\le\hat{\delta}(M,N)\le 1$

(5) $\hat{\delta}(M,N)=\|P_M-Q_N\|$.

Let $A\in B(X,Y)$. If there is $C\in B(Y,X)$ such that $ACA=A$ and $CAC=C$, we call C is a generalized inverse of A and is denoted by A^+_{GI}. In this case, $R(A)$ is closed in Y.

Recall that A is Moore–Penrose invertible, if there is $B\in B(Y,X)$ such that

$$ABA=A,\ BAB=B,\ (AB)^*=AB,\ (BA)^*=BA. \tag{2.1}$$

The operator B in (2.1) is called the Moore–Penrose inverse of A and is denoted as A^+. It is well–known that A is Moore–Penrose invertible iff $R(A)$ is closed in Y. Thus, A is Moore–Penrose invertible iff A^+_{GI} exists.

Let A, $\delta A\in B(X,Y)$ and put $\bar{A}=A+\delta A$. Recall that $\bar{A}$ is the stable perturbation of A if $R(\bar{A})\cap R(A)^\perp=\{0\}$.

The next lemma illustrates some equivalent conditions of the stable perturbation.

Lemma 2.2 ([15, 8]). *Let $A\in B(X,Y)$ with $R(A)$ closed and $\delta A\in B(X,Y)$ with $\|A^+\|\|\delta A\|<1$. Put $\bar{T}=T+\delta T$.*

(*A*) *The following conditions are equivalent.*

(1) $R(\bar{A})\cap R(A)^\perp=\{0\}$

(2) $N(\bar{A})^\perp\cap N(A)=\{0\}$

(3) $R(\bar{A})$ *is closed and* $\bar{A}^+_{GI}=A^+(I+\delta AA^+)^{-1}=(I+A^+\delta A)^{-1}A^+$

(*B*) *If $\bar{A}$ is the stable perturbation of A, then $R(\bar{A})$ is closed and*

$$\|\bar{A}^+\| \leq \frac{\|A^+\|}{1 - \|A^+\|\|\delta A\|}, \ \|\bar{A}^+ - A^+\| \leq \frac{1+\sqrt{5}}{2}\|\bar{A}^+\|\|A^+\|\|\delta A\|.$$

Lemma 2.3. *Let $A \in B(X,Y)$ with $R(A)$ closed. If $Z \in B(Y,X)$ satisfies the conditions: $AZA = A$ and $ZAZ = Z$, then $A^+ = P_{N(A)^\perp} Z P_{R(A)}$.*

Proof. We can check that $P_{N(A)^\perp} Z P_{R(A)}$ satisfies the definition of the Moore–Penrose inverse of A. $\square$

The following result is known when X, Y are all of finite–dimensional (cf. [1]).

Lemma 2.4. *Let $A \in B(X,Y)$ and $T \subset X, S \subset Y$ be closed subspaces. If $A^{(2)}_{T,S}$ exists, then $A^{(2)}_{T,S} = (P_{S^\perp} A P_T)^+$ with $R(A^{(2)}_{T,S}) = T$ and $N(A^{(2)}_{T,S}) = S$.*

Proof. The existence of $A^{(2)}_{T,S}$ implies that $N(A) \cap T = \{0\}$, AT is closed and $Y = AT \dotplus S$. Let $P\colon Y \to S$ be the idempotent operator. Since $R(P) = S$ and $R(I_Y - P) = AT$, it follows that $PP_S = P_S$, $P_S P = P$ and $(I_Y - P)AT = AT$. Noting that

$$\begin{aligned} (I_Y - P)(I_Y - P_S) &= I_Y + PP_S - P_S - P = I_Y - P \\ (I_Y - P_S)(I_Y - P) &= I_Y - P - P_S + P_S P = I_Y - P_S, \end{aligned}$$

we have

$$R(I_Y - P_S) = (I_Y - P_S)(R(I_Y - P)) = (I_Y - P_S)AT = P_{S^\perp} AT$$

and hence $R(P_{S^\perp} A P_T) = R(P_{S^\perp}) = S^\perp$ is closed.

Let $x \in T$ and $P_{S^\perp} Ax = 0$. Then $(I_Y - P)Ax = Ax$, $Ax = P_S Ax$ and hence $0 = PAx = PP_S Ax = P_S Ax = Ax$. Since $N(A) \cap T = \{0\}$, we have $x = 0$ and consequently, $N(P_{S^\perp} A P_T) = T^\perp$. Therefore, $(P_{S^\perp} A P_T)^+$ exists and

$$R((P_{S^\perp} A P_T)^+) = (N(P_{S^\perp} A P_T))^\perp = T \tag{2.2}$$

$$N((P_{S^\perp} A P_T)^+) = (R(P_{S^\perp} A P_T))^\perp = S. \tag{2.3}$$

Since

$$(P_{S^\perp} A P_T)^+ P_{S^\perp} = (P_{S^\perp} A P_T)^+ = P_T (P_{S^\perp} A P_T)^+,$$

by (2.2) and (2.3), it follows that

$$\begin{aligned} (P_{S^\perp} A P_T)^+ &= (P_{S^\perp} A P_T)^+ (P_{S^\perp} A P_T)(P_{S^\perp} A P_T)^+ \\ &= (P_{S^\perp} A P_T)^+ A (P_{S^\perp} A P_T)^+ \end{aligned}$$

and so that $A^{(2)}_{T,S} = (P_{S^\perp} A P_T)^+$. $\square$

Lemma 2.5 ([10, Theorem 11,P100])**.** *Let M be a complemented subspace of H. Let $P \in B(H)$ be an idempotent operator with $R(P) = M$. Let M' be a closed subspace of H satisfying $\hat{\delta}(M, M') < \dfrac{1}{1 + \|P\|}$. Then M' is complemented, that is, $H = R(I - P) \dotplus M'$.*

3 main result

We begin with the key lemma as follows.

Lemma 3.1. *Let $A \in B(X,Y)$. Let $T \subset X$ and $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Let T' be a closed subspace of X such that $\hat{\delta}(T,T') < \dfrac{1}{1+\|A\|\|A^{(2)}_{T,S}\|}$. Then*

$$\hat{\delta}(AT,AT') \le \frac{\|A\|\|A^{(2)}_{T,S}\|\hat{\delta}(T,T')}{1-(1+\|A\|\|A^{(2)}_{T,S}\|)\hat{\delta}(T,T')}.$$

Proof. First we show $\delta(AT,AT') \le \|A\|\|A^{(2)}_{T,S}\|\delta(T,T') \le \|A\|\|A^{(2)}_{T,S}\|\hat{\delta}(T,T')$.

Let $x \in T$. Then $x = A^{(2)}_{T,S}Ax$ and $\|x\| \le \|A^{(2)}_{T,S}\|\|Ax\|$. For any $y \in T'$, we have $\|Ax - Ay\| \le \|A\|\|x-y\|$. So

$$\begin{aligned}
dist(Ax, AT') &= \inf_{y\in T'} \|Ax-Ay\| \le \|A\| \inf_{y\in T'} \|x-y\| \\
&= \|A\| dist(x,T') \le \|A\|\|x\|\delta(T,T') \\
&\le \|A\|\|A^{(2)}_{T,S}\|\|Ax\|\delta(T,T').
\end{aligned}$$

This means that $\delta(AT,AT') \le \|A\|\|A^{(2)}_{T,S}\|\delta(T,T') \le \|A\|\|A^{(2)}_{T,S}\|\hat{\delta}(T,T')$.

Next we show

$$\delta(AT',AT) \le \frac{\|A\|\|A^{(2)}_{T,S}\|\hat{\delta}(T,T')}{1-(1+\|A\|\|A^{(2)}_{T,S}\|)\hat{\delta}(T,T')}$$

when $\hat{\delta}(T,T') < \dfrac{1}{1+\|A\|\|A^{(2)}_{T,S}\|}$.

For $x' \in T'$ and $x \in T$, we have

$$\begin{aligned}
\|Ax'\| &= \|A(x'-x+x)\| \ge \|Ax\| - \|A\|\|x'-x\| \\
&\ge \|A^{(2)}_{T,S}\|^{-1}\|x\| - \|A\|\|x'-x\| \\
&\ge \|A^{(2)}_{T,S}\|^{-1}\|x'\| - \|A^{(2)}_{T,S}\|^{-1}\|x'-x\| - \|A\|\|x'-x\| \\
&\ge \|A^{(2)}_{T,S}\|^{-1}\|x'\| - (\|A^{(2)}_{T,S}\|^{-1} + \|A\|)\|x'-x\|,
\end{aligned}$$

Thus,

$$(\|A^{(2)}_{T,S}\|^{-1} + \|A\|)\|x'-x\| \ge \|A^{(2)}_{T,S}\|^{-1}\|x'\| - \|Ax'\|$$

and consequently,

$$\|A^{(2)}_{T,S}\|^{-1}\|x'\| - \|Ax'\| \le \|x'\|(\|A^{(2)}_{T,S}\|^{-1} + \|A\|)\delta(T',T),$$

that is,

$$\|A^{(2)}_{T,S}\|\|Ax'\| \ge \left[1-(1+\|A\|\|A^{(2)}_{T,S}\|)\delta(T',T)\right]\|x'\|. \tag{3.1}$$

Therefore,

$$\begin{aligned}
dist(Ax', AT) &\le \|A\| dist(x',T) \le \|A\|\|x'\|\delta(T',T) \\
&\le \frac{\|A\|\|Ax'\|\|A^{(2)}_{T,S}\|\hat{\delta}(T,T')}{1-(1+\|A\|\|A^{(2)}_{T,S}\|)\hat{\delta}(T,T')},
\end{aligned}$$

i.e., $\delta(AT', AT) \leq \dfrac{\|A\|\|A^{(2)}_{T,S}\|\hat{\delta}(T,T')}{1-(1+\|A\|\|A^{(2)}_{T,S}\|)\hat{\delta}(T,T')}$ when $\hat{\delta}(T,T') < \dfrac{1}{1+\|A\|\|A^{(2)}_{T,S}\|}$.

The final assertion follows from above arguments. □

Proposition 3.2. *Let $A \in B(X,Y)$ and $T \subset X$, $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Let T' be a closed subspace of X such that $\hat{\delta}(T,T') < \dfrac{1}{(1+\|A\|\|A^{(2)}_{T,S}\|)^2}$. Then $A^{(2)}_{T',S}$ exists and*

(1) $A^{(2)}_{T',S} = P_{T'}(I_X + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'}-P_T))^{-1}A^{(2)}_{T,S}P_{S^\perp}$.

(2) $\|A^{(2)}_{T',S}\| \leq \dfrac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(T,T')}$.

(3) $\|A^{(2)}_{T',S} - A^{(2)}_{T,S}\| \leq \dfrac{1+\sqrt{5}}{2}\|A^{(2)}_{T',S}\|\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(T,T')$.

Proof. By (3.1), $N(A)\cap T' = \{0\}$ when $\hat{\delta}(T,T') < \dfrac{1}{(1+\|A\|\|A^{(2)}_{T,S}\|)^2}$.

Let $P = AA^{(2)}_{T,S}$. Then P is idempotent from Y onto AT along S. By Lemma 3.1, we have

$$\hat{\delta}(AT, AT') \leq \frac{\|A\|\|A^{(2)}_{T,S}\|\hat{\delta}(T,T')}{1-(1+\|A\|\|A^{(2)}_{T,S}\|)\hat{\delta}(T,T')} < \frac{1}{1+\|A\|\|A^{(2)}_{T,S}\|} \leq \frac{1}{1+\|P\|}$$

when $\hat{\delta}(T,T') < \dfrac{1}{(1+\|A\|\|A^{(2)}_{T,S}\|)^2}$. So AT' is complemented and $AT' \dotplus S = Y$ by Lemma 2.5. Therefore. $A^{(2)}_{T',S}$ exists and $A^{(2)}_{T',S} = (P_{S^\perp}AP_{T'})^+$ by Lemma 2.4.

Set $B = P_{S^\perp}AP_T$, $\bar{B} = B + P_{S^\perp}A(P_{T'}-P_T) = P_{S^\perp}AP_{T'}$. Then $N(B^+) = S$ and $R(\bar{B}) = ((N(\bar{B}^+))^\perp = S^\perp$. So $R(\bar{B})\cap N(B^+) = \{0\}$, that is, $\bar{B}$ is the stable perturbation of B.

From Proposition 2.1 (5), we have

$$\|B^+P_{S^\perp}A(P_{T'}-P_T)\| \leq \|A^{(2)}_{T,S}\|\|A\|\|P_{T'}-P_T\| = \|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(T,T') < 1.$$

Hence, by Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} A^{(2)}_{T',S} = \bar{B}^+ &= P_{N(\bar{B})^\perp}(I + B^+P_{S^\perp}A(P_{T'}-P_T))^{-1}B^+P_{R(\bar{B})} \\ &= P_{T'}(I + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'}-P_T))^{-1}A^{(2)}_{T,S}P_{S^\perp}, \end{aligned}$$

$\|A^{(2)}_{T',S}\| \leq \dfrac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(T,T')}$ and

$$\begin{aligned} \|A^{(2)}_{T',S} - A^{(2)}_{T,S}\| &= \|\bar{B}^+ - B^+\| \\ &\leq \frac{1+\sqrt{5}}{2}\|A^{(2)}_{T',S}\|\|A^{(2)}_{T,S}\|\|P_{S^\perp}A(P_{T'}-P_T)\| \\ &\leq \frac{1+\sqrt{5}}{2}\|A^{(2)}_{T',S}\|\|A^{(2)}_{T,S}\|\|A\|\|P_{T'}-P_T\| \\ &= \frac{1+\sqrt{5}}{2}\|A^{(2)}_{T',S}\|\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(T,T'). \end{aligned}$$

□

Similar to Proposition 3.2, we have

Proposition 3.3. *Let $A \in B(X,Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Let $S' \subset Y$ be a closed subspace such that $\hat{\delta}(S,S') < \dfrac{1}{2+\|A\|\|A^{(2)}_{T,S}\|}$. Then $A^{(2)}_{T,S'}$ exists and*

(1) $A^{(2)}_{T,S'} = P_T(I_X + A^{(2)}_{T,S}(P_{(S')^\perp} - P_{S^\perp})AP_T)^{-1}A^{(2)}_{T,S}P_{(S')^\perp}$.

(2) $\|A^{(2)}_{T,S'}\| \le \dfrac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(S,S')}$.

(3) $\|A^{(2)}_{T,S'} - A^{(2)}_{T,S}\| \le \dfrac{1+\sqrt{5}}{2}\|A^{(2)}_{T,S'}\|\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(S,S')$.

Proof. Note that $Q = I_Y - AA^{(2)}_{T,S}$ is an idempotent operator from Y onto S along AT and

$$\hat{\delta}(S,S') < \frac{1}{2+\|A\|\|A^{(2)}_{T,S}\|} \le \frac{1}{1+\|I_Y - Q\|}.$$

So $Y = AT \dot{+} S'$ by Lemma 2.5 and hence $A^{(2)}_{T,S'}$ exists with $A^{(2)}_{T,S'} = (P_{S'^\perp}AP_T)^+$. Using similar methods in the proof of Proposition 3.2, we can get the results. $\square$

Now we present the main result of the paper as follows.

Theorem 3.4. *Let $A \in B(X,Y)$ and let T, $T' \subset X$, S, $S' \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists and* $\max\{\hat{\delta}(T,T'), \hat{\delta}(S,S')\} < \dfrac{1}{(1+\|A\|\|A^{(2)}_{T,S}\|)^2}$. *Then $A^{(2)}_{T',S'}$ exists and*

(1) $A^{(2)}_{T',S'} = P_{T'}\big[I_X + P_{T'}(I + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}A^{(2)}_{T,S}(P_{S^\perp}P_{S'^\perp} - P_{S^\perp})AP_{T'}\big]^{-1}$
$\times P_{T'}(I_X + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}A^{(2)}_{T,S}P_{S^\perp}P_{S'^\perp}$.

(2) $\|A^{(2)}_{T',S'}\| \le \dfrac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|A\|(\hat{\delta}(T,T')+\hat{\delta}(S,S'))}$.

(3) $\|A^{(2)}_{T',S'} - A^{(2)}_{T,S}\| \le \dfrac{1+\sqrt{5}}{2}\dfrac{\|A^{(2)}_{T,S}\|^2\|A\|(\hat{\delta}(T,T')+\hat{\delta}(S,S'))}{1-\|A^{(2)}_{T,S}\|\|A\|(\hat{\delta}(T,T')+\hat{\delta}(S,S'))}$.

Proof. If $\hat{\delta}(T,T') < \dfrac{1}{(1+\|A\|\|A^{(2)}_{T,S}\|)^2}$, then by Proposition 3.2, $A^{(2)}_{T',S}$ exists and

$$A^{(2)}_{T',S} = P_{T'}(I + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}A^{(2)}_{T,S}P_{S^\perp} \tag{3.2}$$

$$\|A^{(2)}_{T',S}\| \le \frac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(T,T')} < \|A^{(2)}_{T,S}\|(1+\|A\|\|A^{(2)}_{T,S}\|) \tag{3.3}$$

for $\hat{\delta}(T,T') < \dfrac{1}{(1+\|A\|\|A^{(2)}_{T,S}\|)^2} \le \dfrac{1}{1+\|A\|\|A^{(2)}_{T,S}\|}$.

Noting that $\|A\|\|A^{(2)}_{T,S}\| \ge \|AA^{(2)}_{T,S}\| \ge 1$ and

$$(1+\|A\|\|A^{(2)}_{T,S}\|)^2 \ge 2 + \|A\|\|A^{(2)}_{T,S}\|(1+\|A\|\|A^{(2)}_{T,S}\|) > 2 + \|A\|\|A^{(2)}_{T',S}\|$$

by (3.3), we have

$$\hat{\delta}(S,S') < \frac{1}{(1+\|A\|\|A^{(2)}_{T,S}\|)^2} < \frac{1}{2+\|A\|\|A^{(2)}_{T',S}\|}.$$

Hence $A^{(2)}_{T',S'}$ exists with $\|A^{(2)}_{T',S'}\| \le \dfrac{\|A^{(2)}_{T',S}\|}{1-\|A^{(2)}_{T',S}\|\|A\|\hat{\delta}(S,S')}$ and

$$A^{(2)}_{T',S'} = P_{T'}(I_X + A^{(2)}_{T',S}(P_{(S')^\perp} - P_{S^\perp})AP_{T'})^{-1}A^{(2)}_{T',S}P_{(S')^\perp}$$

by Proposition 3.3. Thus we have

$$\begin{aligned}A^{(2)}_{T',S'} &= P_{T'}\big[I_X + P_{T'}(I + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'}-P_T))^{-1}A^{(2)}_{T,S}(P_{S^\perp}P_{S'^\perp} - P_{S^\perp})AP_{T'}\big]^{-1}\\ &\quad\times P_{T'}(I + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'}-P_T))^{-1}A^{(2)}_{T,S}P_{S^\perp}P_{S'^\perp}\end{aligned}$$

by (3.2) and

$$\begin{aligned}\|A^{(2)}_{T',S'}\| &\le \frac{1}{1-\frac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(T,T')}\|A\|\hat{\delta}(S,S')} \times \frac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(T,T')}\\ &= \frac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|A\|(\hat{\delta}(T,T')+\hat{\delta}(S,S'))}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|A^{(2)}_{T',S'} - A^{(2)}_{T,S}\| &= \|A^{(2)}_{T',S'} - A^{(2)}_{T',S} + A^{(2)}_{T',S} - A^{(2)}_{T,S}\|\\ &\le \|A^{(2)}_{T',S'} - A^{(2)}_{T',S}\| + \|A^{(2)}_{T',S} - A^{(2)}_{T,S}\|\\ &\le \frac{1+\sqrt{5}}{2}\|A^{(2)}_{T',S}\|\|A\|(\|A^{(2)}_{T',S'}\|\hat{\delta}(S,S') + \|A^{(2)}_{T,S}\|\hat{\delta}(T,T'))\\ &\le \frac{1+\sqrt{5}}{2}\frac{\|A^{(2)}_{T,S}\|\|A\|}{1-\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(T,T')}(\|A^{(2)}_{T',S'}\|\hat{\delta}(S,S') + \|A^{(2)}_{T,S}\|\hat{\delta}(T,T'))\\ &\le \frac{1+\sqrt{5}}{2}\frac{\|A^{(2)}_{T,S}\|\|A\|}{1-\|A^{(2)}_{T,S}\|\|A\|\hat{\delta}(T,T')}\\ &\quad\times\left(\|A^{(2)}_{T,S}\|\hat{\delta}(T,T') + \frac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|A\|(\hat{\delta}(T,T')+\hat{\delta}(S,S'))}\hat{\delta}(S,S')\right)\\ &= \frac{1+\sqrt{5}}{2}\frac{\|A^{(2)}_{T,S}\|^2\|A\|(\hat{\delta}(T,T')+\hat{\delta}(S,S'))}{1-\|A^{(2)}_{T,S}\|\|A\|(\hat{\delta}(T,T')+\hat{\delta}(S,S'))}.\end{aligned}$$

□

Lemma 3.5. *Let A, $\bar{A} = A + E \in B(X,Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Suppose that $\|A^{(2)}_{T,S}\|\|E\| < 1$. Then*

$$\bar{A}^{(2)}_{T,S} = (I_X + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S} = A^{(2)}_{T,S}(I_Y + EA^{(2)}_{T,S})^{-1}.$$

and

$$\|\bar{A}^{(2)}_{T,S}\| \le \frac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|E\|}, \quad \|\bar{A}^{(2)}_{T,S} - A^{(2)}_{T,S}\| \le \frac{\|A^{(2)}_{T,S}\|^2\|E\|}{1-\|A^{(2)}_{T,S}\|\|E\|}.$$

Proof. If $\|A^{(2)}_{T,S}\|\|E\| < 1$, then $I_X + A^{(2)}_{T,S}E$ and $I_Y + EA^{(2)}_{T,S}$ are invertible.

Since $(I_X + A^{(2)}_{T,S}E)A^{(2)}_{T,S} = A^{(2)}_{T,S}(I_Y + EA^{(2)}_{T,S})$, it follows that

$$(I_X + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S} = A^{(2)}_{T,S}(I_Y + EA^{(2)}_{T,S})^{-1}. \tag{3.4}$$

Put $B = (I_X + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S}$. From (3.4), we get that

$$R(B) = R(A^{(2)}_{T,S}) = T, \quad N(B) = N(A^{(2)}_{T,S}) = S, \quad B(A+E)B = B.$$

Therefore, $\bar{A}^{(2)}_{T,S} = (I_X + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S}$ and $\|\bar{A}^{(2)}_{T,S}\| \leq \dfrac{\|A^{(2)}_{T,S}\|}{1 - \|A^{(2)}_{T,S}\|\|E\|}$.

Since

$$\bar{A}^{(2)}_{T,S} - A^{(2)}_{T,S} = (I_X + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S} - A^{(2)}_{T,S} = -(I_X + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S}EA^{(2)}_{T,S},$$

we have

$$\|\bar{A}^{(2)}_{T,S} - A^{(2)}_{T,S}\| \leq \frac{\|A^{(2)}_{T,S}\|^2\|E\|}{1 - \|A^{(2)}_{T,S}\|\|E\|}.$$

$\square$

As an end of this section, we give the perturbation analysis for $A^{(2)}_{T,S}$ when T, S and A all have small perturbation.

Theorem 3.6. *Let A, $\bar{A} = A + E \in B(X,Y)$ and let $T, T' \subset X$, $S, S' \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists and*

$$\max\{\hat{\delta}(T,T'), \hat{\delta}(S,S')\} < \frac{1}{(1 + \|A\|\|A^{(2)}_{T,S}\|)^2}.$$

If $\|A^{(2)}_{T,S}\|\|E\| < \dfrac{1}{1 + \|A^{(2)}_{T,S}\|\|A\|}$, then

(1) $$\begin{aligned}\bar{A}^{(2)}_{T',S'} = \{&I_X + P_{T'}[I_X + P_{T'}(I + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}A^{(2)}_{T,S}\\ &\times (P_{S^\perp}P_{S'^\perp} - P_{S^\perp})AP_{T'}]^{-1}P_{T'}(I_X + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}A^{(2)}_{T,S}\\ &\times P_{S^\perp}P_{S'^\perp}E\}^{-1}P_{T'}\{I_X + P_{T'}(I + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}\\ &\times A^{(2)}_{T,S}(P_{S^\perp}P_{S'^\perp} - P_{S^\perp})AP_{T'}\}^{-1}\\ &\times P_{T'}(I_X + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}A^{(2)}_{T,S}P_{S^\perp}P_{S'^\perp},\end{aligned}$$

(2) $$\|\bar{A}^{(2)}_{T',S'}\| \leq \frac{\|A^{(2)}_{T,S}\|}{1 - \|A^{(2)}_{T,S}\|\left[\|E\| + \|A\|(\hat{\delta}(T,T') + \hat{\delta}(S,S'))\right]},$$

(3) $$\|\bar{A}^{(2)}_{T',S'} - A^{(2)}_{T,S}\| \leq \frac{\|A^{(2)}_{T,S}\|^2\left[\|E\| + \frac{1+\sqrt{5}}{2}\|A\|(\hat{\delta}(T,T') + \hat{\delta}(S,S'))\right]}{1 - \|A^{(2)}_{T,S}\|\left[\|E\| + \|A\|(\hat{\delta}(T,T') + \hat{\delta}(S,S'))\right]}.$$

Proof. $A^{(2)}_{T',S'}$ exists with $\|A^{(2)}_{T',S'}\| \leq \dfrac{\|A^{(2)}_{T,S}\|}{1 - \|A^{(2)}_{T,S}\|\|A\|(\hat{\delta}(T,T') + \hat{\delta}(S,S'))}$ by Theorem 3.4 when $\max\{\hat{\delta}(T,T'), \hat{\delta}(S,S')\} < \dfrac{1}{(1 + \|A\|\|A^{(2)}_{T,S}\|)^2}$. Thus

$$\|A^{(2)}_{T',S'}\|\|E\| \leq \frac{\|E\|\|A^{(2)}_{T,S}\|}{1 - \|A^{(2)}_{T,S}\|\|A\|(\hat{\delta}(T,T') + \hat{\delta}(S,S'))} < \frac{1 + \|A^{(2)}_{T,S}\|\|A\|}{1 + (\|A^{(2)}_{T,S}\|\|A\|)^2} \leq 1,$$

that is, $\|A^{(2)}_{T',S'}\|\|E\| < 1$ by above inequalities for $\|A^{(2)}_{T,S}\|\|A\| \geq \|A^{(2)}_{T,S}A\| \geq 1$. Consequently, $\bar{A}^{(2)}_{T',S'} = (I_X + A^{(2)}_{T',S'}E)^{-1}A^{(2)}_{T',S'}$ by Lemma 3.5. Simple computation shows that

$$\|\bar{A}^{(2)}_{T',S'}\| \leq \frac{\|A^{(2)}_{T,S}\|}{1 - \|A^{(2)}_{T,S}\|\{\|E\| + \|A\|(\hat{\delta}(T,T') + \hat{\delta}(S,S'))\}},$$

$$\begin{aligned}\bar{A}^{(2)}_{T',S'} &= \{I_X + P_{T'}[I_X + P_{T'}(I + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}A^{(2)}_{T,S}(P_{S^\perp}P_{S'^\perp} - P_{S^\perp}) \\ &\quad \times AP_{T'}]^{-1}P_{T'}(I_X + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}A^{(2)}_{T,S}P_{S^\perp}P_{(S')^\perp}E\}^{-1} \\ &\quad \times P_{T'}\{I_X + P_{T'}(I_X + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}A^{(2)}_{T,S}(P_{S^\perp}P_{S'^\perp} - P_{S^\perp}) \\ &\quad \times AP_{T'}\}^{-1}P_{T'}(I_X + A^{(2)}_{T,S}P_{S^\perp}A(P_{T'} - P_T))^{-1}A^{(2)}_{T,S}P_{S^\perp}P_{S'^\perp}.\end{aligned}$$

Noting that

$$\begin{aligned}\bar{A}^{(2)}_{T',S'} - A^{(2)}_{T,S} &= (I_X + A^{(2)}_{T',S'}E)^{-1}A^{(2)}_{T',S'} - A^{(2)}_{T,S} \\ &= (I_X + A^{(2)}_{T',S'}E)^{-1}(A^{(2)}_{T',S'} - (I_X + A^{(2)}_{T',S'}E)A^{(2)}_{T,S}) \\ &= (I_X + A^{(2)}_{T',S'}E)^{-1}(A^{(2)}_{T',S'} - A^{(2)}_{T,S} - A^{(2)}_{T',S'}EA^{(2)}_{T,S}),\end{aligned}$$

we have

$$\begin{aligned}\|\bar{A}^{(2)}_{T',S'} - A^{(2)}_{T,S}\| &\leq \|(I_X + A^{(2)}_{T',S'}E)^{-1}\|(\|A^{(2)}_{T',S'} - A^{(2)}_{T,S}\| + \|A^{(2)}_{T',S'}EA^{(2)}_{T,S}\|) \\ &\leq \frac{1}{1 - \|A^{(2)}_{T',S'}\|\|E\|}(\|A^{(2)}_{T',S'} - A^{(2)}_{T,S}\| + \|A^{(2)}_{T',S'}\|\|E\|\|A^{(2)}_{T,S}\|) \\ &\leq \frac{\|A^{(2)}_{T,S}\|^2\Big[\|E\| + \frac{1+\sqrt{5}}{2}\|A\|(\hat{\delta}(T,T') + \hat{\delta}(S,S'))\Big]}{1 - \|A^{(2)}_{T,S}\|\big[\|E\| + \|A\|(\hat{\delta}(T,T') + \hat{\delta}(S,S'))\big]}.\end{aligned}$$

□

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